Quantum Mechanics, Spin, Lorentz Group

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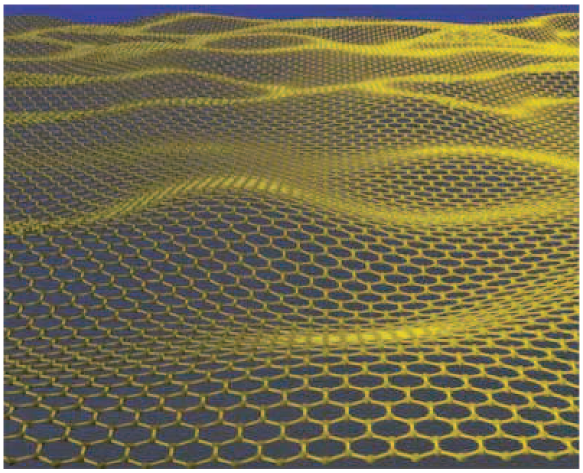

Vector Space

A vector space V over $\mathbb{C}$ is a set which:
1. is a commutative group under addition
2. closed under multiplication by complex numbers such that

$$
\begin{aligned}
&z\,(|v\rangle + |w\rangle) = z\,|v\rangle + z\,|w\rangle\,, \\
&(z_1 + z_2)|v\rangle = z_1|v\rangle + z_2|v\rangle\,, \\
&(z_1 z_2)|v\rangle = z_1(z_2|v\rangle) \\
&1\,|v\rangle = |v\rangle
\end{aligned}
$$

for all $z, z_{1,2} \in \mathbb{C}$ and $|v\rangle, |w\rangle \in V$.

If $\{|1\rangle, |2\rangle, \cdots, |n\rangle\}$ is a basis of V then every $|v\,\rangle \in V$ can be expressed as

$$|v\rangle = v_1|1\rangle + v_2|2\rangle + \cdots + v_n|n\rangle \tag{1}$$

This expression for $|v\rangle$ is unique because of linear independence of the the basis vectors. If the basis is understood we can also write the vector in terms of its coordinates:

$$|v\rangle = v_1|1\rangle + v_2|2\rangle + \cdots + v_n|n\rangle \mapsto \begin{pmatrix} v_1 \\ v_2 \\ . \\ . \\ v_n \end{pmatrix} \tag{2}$$

Linear Transformation $T : V \mapsto W$ is a linear transformation if

$$T(\alpha|v\rangle + \beta|u\rangle) = \alpha\,T(|v\rangle) + \beta\,T(|\,u\rangle) \tag{3}$$
$$\text{for every } |v\rangle, |u\rangle \in V \;\text{ and }\; \alpha, \beta \in \mathbb{C}\,.$$

Dual Space The dual vector space V^* to a vector space V is the set of all linear transformations

$$L : V \mapsto \mathbb{C}$$

Let $\{|1\rangle, |2\rangle, \cdots, |n\rangle\}$ be a basis of V. For $|v\rangle = v_1|1\rangle + v_2|2\rangle + \cdots + v_n|n\rangle$

$$L(|v\rangle) = v_1 L(|1\rangle) + v_2 L(|2\rangle) + \cdots + v_n L(|n\rangle) \tag{4}$$

Thus the action of L on an arbitrary vector is completely determined by the action of L on the basis vectors. Thus n complex numbers $\alpha_i = L(|i\rangle)$ determine the linear transformation completely. We can label the linear transformation L by these numbers so that

$$L_{\alpha_1,\alpha_2,\cdots,\alpha_n}(|i\rangle) = \alpha_i\,, \quad i = 1, 2, \cdots, n\,. \tag{5}$$

It is easy to check that

- $L_{\alpha_1,\alpha_2,\cdots,\alpha_n} + L_{\beta_1,\beta_2,\cdots,\beta_n} = L_{\alpha_1+\beta_1,\alpha_2+\beta_2,\cdots,\alpha_n+\beta_n}$
- $\gamma\, L_{\alpha_1,\alpha_2,\cdots,\alpha_n} = L_{\gamma\alpha_1,\gamma\alpha_2,\cdots,\gamma\alpha_n}$
- $L_{\alpha_1,\alpha_2,\cdots,\alpha_n} = \alpha_1 L_{1,0,0\cdots,0} + \alpha_2 L_{0,1,0,\cdots,0} + \cdots + \alpha_n L_{0,0,0,\cdots,1}$

Thus $V^* = \{L : V \mapsto \mathbb{C} |\, L \text{ is linear}\}$ is a vector space with basis $\{L_{1,0,0\cdots,0}, L_{0,1,0,\cdots,0}, \cdots, L_{0,0,0,\cdots,1}\}$ and hence is of dimension n. Let us denote $L_i = L_{0,0,\cdots,1\,0,\cdots,0}$ (1 is at the i-th place). Then $\{L_1, L_2, \cdots, L_n\}$ is a basis dual to the basis $\{|1\rangle, |2\rangle, \cdots, |n\rangle\}$ of V since

$$L_i(|j\rangle) = \delta_{ij} \tag{6}$$

Since V and V^* have the same dimension they are isomorphic. There is no natural isomorphism. One way of defining an isomorphism is as follows:

$$|v\rangle = v_1|1\rangle + v_2|2\rangle + \cdots + v_n|n\rangle \mapsto L_{|v\rangle} = v_1^* L_1 + v_2^* L_2 + \cdots + v_n^* L_n \tag{7}$$

This isomorphism gives us a Hermitian inner product on V:

$$\begin{aligned} (\cdot,\cdot) \quad &: \quad V \times V \mapsto \mathbb{C} \\ (|v\rangle, |w\rangle) \quad &= \quad L_{|v\rangle}(|w\rangle) = v_1^* w_1 + v_2^* w_2 + \cdots + v_n^* w_n \end{aligned} \tag{8}$$

This inner product satisfies the following properties:

- $(|v\rangle, |w\rangle)^* = (|w\rangle, |v\rangle)$
- $(|v\rangle, a|w\rangle + b|u\rangle) = a\,(|v\rangle, |w\rangle) + b\,(|v\rangle, |u\rangle)$
- $(a|v\rangle + b|u\rangle, |w\rangle) = a^*\,(|v\rangle, |w\rangle) + b^*\,(|v\rangle, |u\rangle)$
- $(|v\rangle, |v\rangle) \geq 0\,, \quad (|v\rangle, |v\rangle) = 0 \;\Leftrightarrow\; |v\rangle = 0$

A vector space with a Hermitian inner product with respect to which it is a complete metric space is called a Hilbert space [1]. $\mathbb{C}^n$ with the Hermitian inner product given above is a complete metric space, hence a Hilbert space, and this is the prototype example of a finite dimensional Hilbert space.

Dirac Notation We will denote the linear transformation $L_{|v\rangle}$ as $\langle v|$. Thus

$$\begin{aligned} (|v\rangle, |w\rangle) \quad &= \quad L_{|v\rangle}(|w\rangle) = \langle v|w\rangle \\ \langle v| \quad &= \quad v_1^*\langle 1| + v_2^*\langle 2| + \cdots + v_n^*\langle n| \end{aligned} \tag{9}$$

[1] A metric space is complete if every Cauchy sequence, in the metric space, converges to a point of the metric space.

Adjoint If $\widehat{A} : V \mapsto V$ is a linear operator then the adjoint of $\widehat{A}$ denoted by $\widehat{A}^\dagger$ is a linear operator $\widehat{A}^\dagger : V \mapsto V$ defined by the relation

$$(|v\rangle, \widehat{A}|w\rangle) = (\widehat{A}^\dagger|v\rangle, |w\rangle) \tag{10}$$

- $(\widehat{A}^\dagger)^\dagger = \widehat{A}$
- $(\widehat{A} + \widehat{B})^\dagger = \widehat{A}^\dagger + \widehat{B}^\dagger$
- $(\lambda\,\widehat{A})^\dagger = \lambda^*\,\widehat{A}^\dagger\,,\ \lambda \in \mathbb{C}$
- $(\widehat{A}\,\widehat{B})^\dagger = \widehat{B}^\dagger\widehat{A}^\dagger$

$\widehat{A}$ is called Hermitian if $\widehat{A}^\dagger = \widehat{A}$
$\widehat{A}$ is called Unitary if $\widehat{A}^\dagger\widehat{A} = \widehat{A}\widehat{A}^\dagger = \mathbb{1}$

Matrix Representation of Operators Let $\mathcal{H}$ be a Hilbert space with an orthonormal basis $\{|1\rangle, |2\rangle, \cdots, |n\rangle\}$

$$\langle\, i \,|\, j \,\rangle = \delta_{ij}$$

$\{\langle\, 1\,|, \langle\, 2\,|, \cdots, \langle\, n\,|\}$ being the basis of the dual space:

$$\langle i| : \mathcal{H} \mapsto \mathbb{C}$$

A linear operator $\widehat{A} : \mathcal{H} \mapsto \mathcal{H}$ is completely determined by its action on the basis of $\mathcal{H}$

$$\widehat{A}\,|\, j\,\rangle = \sum_{i=1}^{n} A_{ij}|\, i\,\rangle\,,\ \ A_{ij} \in \mathbb{C}\,.$$

The numbers A_{ij} form an $n \times n$ matrix which is the matrix representation of the operator $\widehat{A}$ in the basis $\{|1\rangle, |2\rangle, \cdots, |n\rangle\}$. The numbers A_{ij} are also called the matrix elements of the operator $\widehat{A}$:

$$\begin{aligned}
\langle\, i\,|\widehat{A}|\, j\,\rangle &= \langle\, i\,|\Big(\sum_k A_{kj}|\, k\,\rangle\Big) = \sum_k \langle\, i\,|A_{kj}|, k\,\rangle \\
&= \sum_k A_{kj}\langle\, i\,|\, k\,\rangle = \sum_k A_{kj}\delta_{ik} \qquad (11)\\
&= A_{ij} \qquad (12)
\end{aligned}$$

$$|v\rangle = v_1|1\rangle + v_2|2\rangle + \cdots + v_n|n\rangle \mapsto \begin{pmatrix} v_1 \\ v_2 \\ . \\ . \\ v_n \end{pmatrix}, \; v_i = \langle\, i \,|v\rangle \in \mathbb{C}$$

$$\widehat{A}|v\rangle = |w\rangle \mapsto \begin{pmatrix} A_{11} & A_{12} & . & . & A_{1n} \\ A_{21} & . & . & . & A_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ A_{n1} & A_{2n} & . & . & A_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ . \\ . \\ v_n \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ . \\ . \\ w_n \end{pmatrix}$$

The linearity property of the linear operator allows us to express an arbitrary linear operator as linear combination of operators that map a basis state to a basis state. The operator that maps a basis state $|\, j\,\rangle$ to a basis state $|\, i\,\rangle$ is given by $|\, i\,\rangle\langle\, j\,|$:

$$|\, i\,\rangle\langle\, j\,| : \mathcal{H} \mapsto \mathcal{H}$$
$$\Big(|\, i\,\rangle\langle\, j\,|\Big)|v\rangle = |\, i\,\rangle\, v_j \,.$$

There are n^2 such operators and in a given basis an operator $\widehat{A}$ is determined by n^2 numbers A_{ij}. Thus we can write

$$\widehat{A} = \sum_{i,j=1}^{n} \alpha_{ij}|\, i\,\rangle\langle\, j\,|, \;\; \alpha_{ij} \in \mathbb{C} \tag{13}$$

$$\begin{aligned} A_{ij} &= \langle\, i\,|\widehat{A}|\, j\,\rangle = \langle\, i\,|\Big(\sum_{k,l=1}^{n} \alpha_{kl}|\, k\,\rangle\langle\, l\,|\Big)|\, j\,\rangle = \sum_{k,l}^{n} \alpha_{kl}\langle\, i\,|\, k\,\rangle\langle\, l\,|\, j\,\rangle \\ &= \sum_{k,l}^{n} \alpha_{kl}\delta_{ik}\delta_{lj} = \alpha_{ij} \end{aligned} \tag{14}$$

Thus

$$\boxed{\widehat{A} = \textstyle\sum_{i,j}^{n} A_{ij}|\, i\,\rangle\langle\, j\,|} \tag{15}$$

If $A_{ij} = \delta_{ij}$ we get the identity operator thus

$$\boxed{\mathbb{1} = \textstyle\sum_{i} |\, i\,\rangle\langle\, i\,|} \tag{16}$$

If $\widehat{A}$ and $\widehat{B}$ are two linear operators then

$$(AB)_{ij} = \langle\, i\,|\widehat{A}\widehat{B}|\, j\,\rangle = \langle\, i\,|\widehat{A}\mathbb{1}\widehat{B}|\, j\,\rangle = \langle\, i\,|\widehat{A}\Big(\sum_{k} |\, k\,\rangle\langle\, k\,|\Big)\widehat{B}|\, j\,\rangle = \sum_{k} A_{ik}B_{kj} \tag{17}$$

Thus the matrix representation of product of two operators is the product of matrix representations of the two operators.

Let us see what the adjoint of $|i\rangle\langle j|$ is:

$$\begin{aligned} \Big(|v\rangle, (|i\rangle\langle j|)|w\rangle\Big) &= \langle v|(|i\rangle\langle j|)|w\rangle = \langle v|i\rangle\,\langle j|w\rangle = v_i^* w_j = v_i^*\Big(|j\rangle, |w\rangle\Big) = \Big(v_i\,|j\rangle, |w\rangle\Big) \\ &= \Big((|j\rangle\langle i|)|v\rangle, |w\rangle\Big) \end{aligned} \tag{18}$$

Thus it follows that

$$\boxed{\Big(|i\rangle\langle j|\Big)^\dagger = |j\rangle\langle i|} \tag{19}$$

From Eq(15) and Eq(19) we get

$$\widehat{A}^\dagger = \sum_{i,j} A_{ij}^* |j\rangle\langle i| = \sum_{i,j} A_{ji}^* |i\rangle\langle j| \tag{20}$$

Thus the matrix of operator $\widehat{A}^\dagger$ is the adjoint (complex conjugate, transpose) of the matrix of operator $\widehat{A}$.

Eigenvalues and Eigenvectors of a Hermitian operator

Let $\widehat{H}$ be a Hermitian operator with eigenvector $|v\rangle$ and corresponding eigenvalue λ:

$$\widehat{H}|v\rangle = \lambda\,|v\rangle \implies \langle v|\widehat{H} = \lambda^*\,\langle v| \tag{21}$$

Since $\widehat{H}^\dagger = \widehat{H}$. Thus the matrix element $\langle v|\widehat{H}|v\rangle$ can be evaluated in two different ways

$$\langle v|\widehat{H}|v\rangle = \begin{cases} \lambda\langle v|v\rangle \\ \lambda^*\langle v|v\rangle \end{cases} \implies \lambda = \lambda^* \tag{22}$$

Thus a Hermitian operator has real eigenvalues

Let $|v\rangle$ and $|w\rangle$ be two eigenvectors of $\widehat{H}$ with eigenvalues λ and μ respectively then the matrix element $\langle w|\widehat{H}|v\rangle$ can be evaluated in two different ways

$$\langle w|\widehat{H}|v\rangle = \begin{cases} \lambda\langle w|v\rangle \\ \mu\langle w|v\rangle \end{cases} \implies (\lambda - \mu)\langle w|v\rangle = 0 \tag{23}$$

Thus the eigenvectors of a Hermitian operator with distinct eigenvalues are orthogonal

Eigenvalues and Eigenvectors of a Unitary operator

Let $\widehat{U}$ be a unitary operator with eigenvector $|v\rangle$ and corresponding eigenvalue λ:

$$\widehat{U}|v\rangle = \lambda\,|v\rangle \implies \langle v|\widehat{U}^\dagger = \lambda^*\,\langle v| \tag{24}$$

Since $\widehat{U}^\dagger\widehat{U} = \mathbb{1}$. Thus the matrix element $\langle v|\widehat{H}|v\rangle$ can be evaluated in two different ways

$$\begin{aligned}\langle v|v\rangle &= \langle v|\widehat{U}^\dagger\widehat{U}|v\rangle \\ &= \langle v|\lambda^*\,\lambda|v\rangle = |\lambda|^2\langle v|v\rangle \implies |\lambda| = 1\end{aligned} \tag{25}$$

Thus a unitary operator has eigenvalues which are of the type $e^{i\theta}$

Tensor Product

Given two vector spaces V and W we can form a new vector space called the tensor product of V and W denoted by $V\otimes W$. If $\{|v_1\rangle,\cdots,|v_n\rangle\}$ is a basis of V and $\{|w_1\rangle,\cdots,|w_m\rangle\}$ is a basis of W then a basis of $V\otimes W$ is given by[2]

$$\{|v_1\rangle\otimes|w_1\rangle, |v_1\rangle|w_2\rangle\cdots|v_n\rangle\otimes|w_{m-1}\rangle, |v_n\rangle\otimes|w_m\rangle\} \tag{26}$$

thus

$$\text{dim}(V\otimes W) = \text{dim}(V)\text{dim}(W)$$

If $L: V\mapsto V$ and $K: W\mapsto W$ are linear transformations then $L\otimes K: V\otimes W\mapsto V\otimes W$ is a linear transformation defined by

$$L\otimes K\,(|v\rangle\otimes|w\rangle) = L|v\rangle\otimes K|w\rangle$$

In the above basis L is a $n\times n$ matrix and K is a $m\times m$ matrix whose entries we will denote by $L_{i\,j}$ and $K_{a\,b}$ respectively $(i,j = 1,2,\cdots n\&a,b = 1,2,\cdots m)$. The linear transformation $L\otimes K$ (in the basis 26) is a $nm\times nm$ matrix given by

$$\begin{pmatrix} L_{1\,1}K & L_{1\,2}K & \cdots & L_{1\,n}K \\ L_{2\,1}K & L_{2\,2}K & \cdots & L_{2\,n}K \\ \vdots & \vdots & \vdots & \vdots \\ L_{n\,1}K & L_{n\,2}K & \cdots & L_{n\,n}K \end{pmatrix} \text{ where } \quad L_{ij}K := \begin{pmatrix} L_{i\,j}K_{1\,1} & L_{i\,j}K_{1\,2} & \cdots & L_{i\,j}K_{1\,m} \\ L_{i\,j}K_{2\,1} & L_{i\,j}K_{2\,2} & \cdots & L_{i\,j}K_{2\,m} \\ \vdots & \vdots & \vdots & \vdots \\ L_{i\,j}K_{m\,1} & L_{i\,j}K_{m\,2} & \cdots & L_{i\,j}K_{m\,m} \end{pmatrix} \tag{27}$$

$|x\rangle = \sum_{i=1}^n x^i|v_i\rangle$ and $|y\rangle = \sum_{a=1}^m y^a|w_a\rangle$ then

$$|x\rangle\otimes|y\rangle = \sum_{i,a} x^i y^a|v_i\rangle\otimes|w_a\rangle \tag{28}$$

[2] $n = \text{dim}V,\, m = \text{dim}W$

This implies that

$$|x\rangle \mapsto \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix}, |y\rangle \mapsto \begin{pmatrix} y^1 \\ y^2 \\ \vdots \\ y^m \end{pmatrix} \quad \text{then} \tag{29}$$

$$|x\rangle \otimes |y\rangle \mapsto \begin{pmatrix} x^1 y^1 \\ x^1 y^2 \\ \vdots \\ x^n y^{m-1} \\ x^n y^m \end{pmatrix}$$

Exercise 1: Let $\mathcal{H}$ be a three dimensional Hilbert space with an orthonormal basis $\{|1\rangle, |2\rangle, |3\rangle\}$ (Orthonormal means that the basis vectors are unit vectors and orthogonal to each other i.e., $\langle\, i \,|\, j \,\rangle = \delta_{ij}$).

a. Find the matrix representation of the following operators:

$$\begin{aligned} &|1\rangle\langle 1| \\ &|2\rangle\langle 2| \\ &|3\rangle\langle 3| \\ &|1\rangle\langle 2| - |2\rangle\langle 1| \\ &|1\rangle\langle 3| - |2\rangle\langle 3| + |2\rangle\langle 1| \\ &2|1\rangle\langle 1| - \frac{1}{\sqrt{2}}|3\rangle\langle 2| + i|2\rangle\langle 2| \end{aligned}$$

b. Which of the above operators are Hermitian.

c. Let $\widehat{A} = i|1\rangle\langle 2| - i|2\rangle\langle 1| + |3\rangle\langle 3|$. Show that $\widehat{A}$ is Hermitian and find the eigenvalues and eigenstates of $\widehat{A}$.

d. Show that the eigenstates of $\widehat{A}$ are orthogonal.

e. Find the matrix representation of the operators given in part **a** in the basis of normalized eigenstates of $\widehat{A}$.

Let $\mathcal{H}$ be a two dimensional Hilbert space with orthonormal basis $\{|1\rangle, |2\rangle\}$. Let $\widehat{H}$ be the Hamiltonian of the system. In the basis $\{|1\rangle, |2\rangle\}$ the Hamiltonian is a 2×2 Hermitian matrix:

$$\widehat{H} \xrightarrow{\{|1\rangle,|2\rangle\}} H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} , \quad H_{21} = H_{12}^* , \;\; H_{11}, H_{22} \text{ are real} . \tag{30}$$

In general this is not a diagonal matrix and therefore the basis states $|1\rangle$ and $|2\rangle$ are not the eigenstates of the Hamiltonian. We denote the eigenstates of the Hamiltonian by $|+\rangle$ and $|-\rangle$. These are the stationary states of the system. When a measurement of the energy of the system is made it will be found in one the stationary states with energy equal to the corresponding eigenvalue of the stationary state. Thus possible energies of the system are the eigenvalues given by

$$E_\pm = \frac{H_{11} + H_{22} \pm \sqrt{(H_{11} - H_{22})^2 + 4H_{12}H_{21}}}{2} \tag{31}$$

The Hermitian matrix H has four real numbers in it. We can write this matrix as a linear combination (with real coefficients) of four Hermitian matrices (which form a basis of space of Hermitian matrices). These four matrices are $\mathbb{1}, \sigma_1, \sigma_2, \sigma_3$. The matrices $\sigma_{1,2,3}$ are called Pauli matrices and are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{32}$$

The Pauli matrices satisfy some very interesting properties which are very useful in calculations:

$$\begin{aligned}
&\sigma_a^2 = \mathbb{1} \\
&\sigma_1\sigma_2\sigma_3 = i\mathbb{1} \\
&\{\sigma_a, \sigma_b\} = 2\delta_{ab} \\
&\text{Tr}(\sigma_a) = 0 \\
&\sigma_a\sigma_b = i\sigma_c , \;\; \text{where } a, b, c \text{ is a cyclic permutation of } 1, 2, 3 \\
&[\sigma_a, \sigma_b] = 2i\epsilon_{abc}\sigma_c \text{ (summation over } c) \\
&\sigma_a\sigma_b = \delta_{ab} + i\epsilon_{abc}\sigma_c \text{ (summation over } c) \\
&\text{Define } \vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \text{ then } (\vec{n}\cdot\vec{\sigma})^2 = (\vec{n}\cdot\vec{n})\,\mathbb{1} \\
&(\vec{n}\cdot\vec{\sigma})(\vec{m}\cdot\vec{\sigma}) = (\vec{n}\cdot\vec{m})\mathbb{1} + i\vec{\sigma}\cdot\,(\vec{n}\times\vec{m}) \\
&\exp\Big(i\theta\hat{n}\cdot\sigma\Big) = \cos(\theta) + i\hat{n}\cdot\vec{\sigma}\sin(\theta) , \;\; \hat{n} \text{ is a unit vector}
\end{aligned} \tag{33}$$

The Hamiltonian can be written as a linear combination of the identity matrix and the three Pauli matrices:

$$\begin{aligned} \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} &= n_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + n_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + n_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + n_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ H &= n_0 \mathbb{1} + n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3 = n_0\mathbb{1} + \vec{n}\cdot\vec{\sigma} \\ n_0 &= \frac{1}{2}\text{Tr}H = \frac{H_{11}+H_{22}}{2}, \quad \vec{n} = \frac{1}{2}\text{Tr}(H\vec{\sigma}) = \frac{1}{2}\Big(H_{12}+H_{21}, i(H_{12}-H_{21}), H_{11}-H_{22}\Big) \end{aligned} \tag{34}$$

Since $(\vec{n}\cdot\vec{\sigma})^2 = (\vec{n}\cdot\vec{n})\,\mathbb{1}$ therefore the eigenvalues of $\vec{n}\cdot\sigma$ are $\pm|\vec{n}|$. This implies that the eigenvalues of the Hamiltonian H are:

$$E_\pm = n_0 \pm |\vec{n}| \tag{35}$$

The eigenvectors of H will also be eigenvectors of $\vec{n}\cdot\vec{\sigma}$ since every vector is an eigenvector of $\mathbb{1}$:

$$\begin{aligned} \vec{n}\cdot\vec{\sigma} &= \begin{pmatrix} n_3 & n_1 - in_2 \\ n_1 + in_2 & -n_3 \end{pmatrix} \\ \begin{pmatrix} n_3 & n_1 - in_2 \\ n_1 + in_2 & -n_3 \end{pmatrix}\begin{pmatrix} \alpha_\pm \\ \beta_\pm \end{pmatrix} &= (E_\pm - n_0)\begin{pmatrix} \alpha_\pm \\ \beta_\pm \end{pmatrix} \end{aligned} \tag{36}$$

Solving the above linear equations we get the following two eigenvectors of H:

$$\begin{pmatrix} \alpha_\pm \\ \beta_\pm \end{pmatrix} = \begin{pmatrix} \pm(n_1 - in_2) \\ |\vec{n}| \mp n_3 \end{pmatrix}, \quad \text{with eigenvalue } E_\pm = n_0 \pm |\vec{n}|\,. \tag{37}$$

These eigenvectors are not normalized so are not unit vectors. Since $\vec{n}$ is a set of three real numbers we can express these numbers in terms of spherical polar coordinates

$$\begin{aligned} \vec{n} &= (n_1, n_2, n_3) = |\vec{n}|(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\,) \\ |\vec{n}| &= \sqrt{(H_{11}-H_{22})^2 + H_{12}H_{21}}\,,\ \cos\theta = \frac{H_{11}-H_{22}}{2\sqrt{(H_{11}-H_{22})^2 + H_{12}H_{21}}}\,,\ \cos\phi = \frac{H_{12}+H_{21}}{2|H_{12}|} \end{aligned} \tag{38}$$

In terms of the spherical polar coordinates (θ, ϕ) the normalized eigenvectors of H are given by

$$\begin{aligned} &\begin{pmatrix} \cos(\frac{\theta}{2}) \\ e^{i\phi}\sin(\frac{\theta}{2}) \end{pmatrix}, \quad \text{with eigenvalue } E_+ = n_0 + |\vec{n}|\,, \\ &\begin{pmatrix} e^{-i\phi}\sin(\frac{\theta}{2}) \\ -\cos(\frac{\theta}{2}) \end{pmatrix}, \quad \text{with eigenvalue } E_- = n_0 - |\vec{n}|\,, \end{aligned} \tag{39}$$

Since $1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $2\rangle == \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ therefore the eigenstates of $\widehat{H}$ are:

$$\begin{aligned} |+\rangle &= \cos\left(\frac{\theta}{2}\right)|1\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|2\rangle \\ |-\rangle &= e^{-i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle - \cos\left(\frac{\theta}{2}\right)|2\rangle \end{aligned} \tag{40}$$

Since $\langle i|j\rangle = \delta_{ij}\,, i, j = 1, 2$ therefore it follows that: $\langle +|+\rangle = \langle -|-\rangle = 1$ and $\langle +|-\rangle = \langle -|+\rangle = 0$ and

$$\begin{aligned} |1\rangle &= \cos\left(\frac{\theta}{2}\right)|+\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|-\rangle \\ |2\rangle &= e^{-i\phi}\sin\left(\frac{\theta}{2}\right)|+\rangle - e^{i\phi}\cos\left(\frac{\theta}{2}\right)|-\rangle \end{aligned} \tag{41}$$

Exercise 2: A spin-$\frac{1}{2}$ is also a two state system. The spin can be either parallel to the z-axis or anti-parallel to it. We denote these twos states by $|+\rangle$ and $|-\rangle$. The magnetic moment for a spin-$\frac{1}{2}$ particle is given by $\vec{\mu} = \frac{1}{2}\gamma\,\hbar\,\vec{\sigma}$. γ is the gyromagnetic ratio. If such a particle is placed in a constant magnetic field $\vec{B}_0$ the Hamiltonian of the system is given by $\widehat{H}_0 = -\vec{\mu}\cdot\vec{B}$. Suppose that the magnetic field is in the z-direction so that the matrix representation of the Hamiltonian (in the basis $|+\rangle, |-\rangle$) is given by

$$H_0 = -\frac{\gamma\,\hbar\,B_0}{2}\sigma_3 = -\frac{\gamma\,\hbar\,B_0}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

If we now perturb the system by introducing a time varying magnetic field $\vec{B}_1(t) = B_1(\cos(\omega\,t)\vec{\mathbf{i}} - \sin(\omega\,t)\vec{\mathbf{j}})$. The matrix representation of the new Hamiltonian in the basis $|+\rangle, |-\rangle$ is given by

$$H = H_0 - \vec{\mu}\cdot\,\vec{B}_1(t) = -\frac{\hbar}{2}\begin{pmatrix} \omega_0 & \omega_1 e^{i\omega\,t} \\ \omega_1\,e^{-\omega\,t} & \omega_0 \end{pmatrix} \tag{42}$$

where $\omega_1 = \gamma\,B_1$ is called the Rabi frequency. We will calculate the probability of transition from state $-\rangle$ to the state $|+\rangle$.

a. Express the Hamiltonian $\widehat{H}$ as linear combination of operators $|+\rangle\langle +|, |+\rangle\langle -|, |-\rangle\langle +|, |-\rangle\rangle -|$.

a. Arbitrary state at time t is given by $|\psi(t)\rangle = c_+(t)|+\rangle + c_-(t)|-\rangle$ where $|c_+(t)|^2 + |c_-(t)|^2 = 1$. Using the Schrödinger equation find the coupled differential equations satisfied by the coefficients $c_\pm(t)$.

b. Using the ansatz $c_\pm(t) = \gamma_\pm(t)\,e^{-i\lambda_\pm\,t/\hbar}$ obtain the differential equation satisfied by $\gamma_\pm(t)$. $\Big(\lambda_\pm$ are the eigenvalues of $\widehat{H}_0$. In the absence of $\widehat{H}_1$ $\gamma_\pm(t)$ would be independent of time $t.\Big)$

c. Convert the set of coupled first order differential equations satisfied by $\gamma_\pm(t)$ into a set of uncoupled second order differential equations.
d. Solve the second order differential equations satisfied by $\gamma_\pm(t)$ subject to the initial condition $\gamma_+(0) = 0, \gamma_-(0) = 1$.
e. Show that the probability of transition from state $|-\rangle$ at time $t = 0$ to the state $|+\rangle$ at time t is given by

$$P(t) = \frac{\omega_1^2}{\omega_1^2 + (\omega - \omega_0)^2} \sin^2\Big(\frac{\sqrt{\omega_1^2 + (\omega - \omega_0)^2}\, t}{2}\Big)$$

In classical mechanics the angular momentum is given by

$$\vec{L} = \vec{r} \times \vec{p} \tag{43}$$

In quantum mechanics angular momentum is an observable hence there is a corresponding operator

$$\vec{\widehat{L}} = (\widehat{L}_1, \widehat{L}_2, \widehat{L}_3) = (\widehat{x}_2\,\widehat{p}_3 - \widehat{x}_3\,\widehat{p}_2, \widehat{x}_3\,\widehat{p}_1 - \widehat{x}_1\,\widehat{p}_3, \widehat{x}_1\,\widehat{p}_2 - \widehat{x}_2\,\widehat{p}_1) \tag{44}$$

The position and momentum operators satisfy non trivial commutation relations

$$[\widehat{x}_a, \widehat{x}_b] = 0\,, \;\; [\widehat{p}_a, \widehat{p}_b] = 0\,, \;\; [\widehat{x}_a, \widehat{p}_b] = i\hbar\,\delta_{ab} \tag{45}$$

therefore it follows that

$$[\widehat{L}_a, \widehat{L}_b] \;=\; i\hbar \sum_{c=1}^{3} \epsilon_{abc} \widehat{L}_c \tag{46}$$

The only non-zero values of ϵ_{abc} are: $\epsilon_{123} = \epsilon_{312} = \epsilon_{231} = 1$

$$\epsilon_{132} \;=\; \epsilon_{321} = \epsilon_{213} = -1$$

The eigenvalues of these operators $\widehat{L}_a$ have dimensions same as that of $\hbar$ so we can defined operators $\widehat{J}_a$ whose eigenvalues are dimensionless by

$$\begin{aligned} \widehat{J}_a &= \frac{\widehat{L}_a}{\hbar} \\ [\widehat{J}_a, \widehat{J}_b] &= i \sum_{c=1}^{3} \epsilon_{abc} \widehat{J}_c \end{aligned} \tag{47}$$

From the commutation relation it follows that the operator $\widehat{J}^2 = \widehat{J}_1^2 + \widehat{J}_2^2 + \widehat{J}_3^2$ commutes with $\widehat{J}_a\,,\;\; a = 1, 2, 3$,

$$[\widehat{J}^2, \widehat{J}_a] = 0\,, \;\; a = 1, 2, 3\,. \tag{48}$$

Thus the we can pick a basis of the Hilbert space $\mathcal{H}$ in which the operator $\widehat{J}^2$ is diagonal. Since $\widehat{J}_3$ also commutes with $\widehat{J}^2$ therefore we can pick the vectors in the basis of $\mathcal{H}$ to be eigenvectors of both $\widehat{J}^2$ and $\widehat{J}_3$. We label the basis vectors with the eigenvalues of $\widehat{J}^2$ and $\widehat{J}_3$ (which are real since these are Hermitian operators) and take these vectors to be orthonormal:

$$\begin{aligned} \widehat{J}^2|\lambda, m\rangle &= \lambda^2\,|\lambda, m\rangle \\ \widehat{J}_3|\lambda, m\rangle &= m\,|\lambda, m\rangle \\ \langle\lambda, m|\lambda', m'\rangle &= \delta_{\lambda,\lambda'}\,\delta_{m,m'}\,. \end{aligned} \tag{49}$$

The eigenvalue of $\widehat{J}^2$ is positive (and therefore we have written it as λ^2) since

$$\langle\lambda, m|\widehat{J}^2|\lambda, m\rangle = ||\widehat{J}_1|\lambda, m\rangle||^2 + ||\widehat{J}_2|\lambda, m\rangle||^2 + ||\widehat{J}_3|\lambda, m\rangle||^2 \tag{50}$$

We will determine these eigenvalues λ and m.

Let us define $\widehat{J}_\pm = \widehat{J}_1 \pm i\widehat{J}_2$ then

$$\begin{aligned} [\widehat{J}^2, \widehat{J}_\pm] &= 0\,,\ [\widehat{J}_3, \widehat{J}_\pm] = \pm\widehat{J}_\pm \\ \widehat{J}_+\widehat{J}_- &= (\widehat{J}_1 + i\widehat{J}_2)(\widehat{J}_1 - i\widehat{J}_2) = \widehat{J}_1^2 + \widehat{J}_2^2 - i[\widehat{J}_1, \widehat{J}_2] = \widehat{J}^2 - \widehat{J}_3^2 + \widehat{J}_3 \\ \widehat{J}_-\widehat{J}_+ &= \widehat{J}^2 - \widehat{J}_3^2 - \widehat{J}_3 \end{aligned} \tag{51}$$

$\widehat{J}_\pm$ have been defined since there action on $|\lambda, m\rangle$ is very interesting:

$$\begin{aligned} \widehat{J}_3\widehat{J}_\pm|\lambda, m\rangle &= (\widehat{J}_\pm\widehat{J}_3 \pm \widehat{J}_\pm)|\lambda, m\rangle = (m \pm 1)\widehat{J}_\pm|\lambda, m\rangle \\ \widehat{J}_\pm|\lambda, m\rangle &\propto |\lambda, m \pm 1\rangle\,. \end{aligned} \tag{52}$$

From the definition of these new operators it is clear that $\widehat{J}_+ = \widehat{J}_-^\dagger$. Therefore

$$\begin{aligned} ||\widehat{J}_+|\lambda, m\rangle||^2 &= \langle\,\lambda, m|\widehat{J}_-\widehat{J}_+|\lambda, m\rangle = \langle\,\lambda, m|\Big(\widehat{J}^2 - \widehat{J}_3^2 - \widehat{J}_3\Big)|\lambda, m\rangle \\ &= \langle\,\lambda, m|\widehat{J}^2|\lambda, m\rangle - \langle\,\lambda, m|\widehat{J}_3^2|\lambda, m\rangle - \langle\,\lambda, m|\widehat{J}_3|\lambda, m\rangle \\ &= \lambda^2 - m^2 + m \geq 0 \end{aligned} \tag{53}$$

and similarly

$$\begin{aligned} ||\widehat{J}_-|\lambda, m\rangle||^2 &= \langle\,\lambda, m|\widehat{J}_+\widehat{J}_-|\lambda, m\rangle = \langle\,\lambda, m|\Big(\widehat{J}^2 - \widehat{J}_3^2 + \widehat{J}_3\Big)|\lambda, m\rangle \\ &= \langle\,\lambda, m|\widehat{J}^2|\lambda, m\rangle - \langle\,\lambda, m|\widehat{J}_3^2|\lambda, m\rangle + \langle\,\lambda, m|\widehat{J}_3|\lambda, m\rangle \\ &= \lambda^2 - m^2 - m \geq 0 \end{aligned} \tag{54}$$

From the above two inequalities we see that

$$\lambda^2 \geq m^2 \implies -\lambda \leq m \leq +\lambda \tag{55}$$

Thus for a given eigenvalues of $\widehat{J}^2$ the eigenvalues of $\widehat{J}_3$ are bounded. Hence for a given eigenvalue λ^2 of $\widehat{J}^2$ we have a maximum and minimum eigenvalues of $\widehat{J}_3$ which we will denote by m_{max} and m_{min} respectively.

$$\begin{aligned} \widehat{J}_+|\lambda, m_{max}\rangle = 0 \\ \widehat{J}_-|\lambda, m_{min}\rangle = 0 \end{aligned} \tag{56}$$

From Eq(53) and Eq(54) it follows that

$$\lambda^2 = \begin{cases} m_{max}^2 + m_{max} \\ m_{min}^2 - m_{max} \end{cases} \implies m_{max} = -m_{min} \tag{57}$$

Since $\widehat{J}_+$ takes us from m_{min} to m_{max} in steps of one therefore

$$m_{max} - m_{min} = 2m_{max} \in \mathbb{Z}_{\geq 0}\,, \tag{58}$$

Let us call $m_{max} = j$ then

$$\lambda^2 = j(j+1)\,, \quad -j \leq m \leq +j \tag{59}$$

$$\begin{aligned} \widehat{J}_+|j,m\rangle &= C^+_{j,m}|j,m+1\rangle\,, \qquad \widehat{J}_-|j,m\rangle = C^-_{j,m}|j,m-1\rangle \\ |C^+_{j,m}|^2 &= ||\widehat{J}_+|j,m\rangle||^2 = \langle j,m|\widehat{J}_-\widehat{J}_+|j,m\rangle = j(j+1) - m^2 - m \\ |C^-_{j,m}|^2 &= ||\widehat{J}_-|j,m\rangle||^2 = \langle j,m|\widehat{J}_+\widehat{J}_-|j,m\rangle = j(j+1) - m^2 + m \end{aligned} \tag{60}$$

Thus the Hilbert spaces $\mathcal{H}$ is a direct sum

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_0 \oplus \mathcal{H}_{\frac{1}{2}} \oplus \mathcal{H}_1 \oplus \mathcal{H}_{\frac{3}{2}} \oplus \cdots \\ \mathcal{H}_j & \quad \text{is spanned by } \{|j,-j\rangle, |j,-j+1\rangle, |j,-j+2\rangle, \cdots, |j,j-1\rangle, |j,j\rangle\} \\ \dim_{\mathbb{C}}\mathcal{H}_j &= 2j+1 \end{aligned} \tag{61}$$

$\underline{\mathcal{H}_0}$:

It is one dimensional with basis vector $|0,0\rangle$. Thus the operators are all numbers (1×1 matrices):

$$\widehat{J}^2 \xrightarrow{\{|0,0\rangle\}} \begin{pmatrix} 0 \end{pmatrix}\,, \ \widehat{J}_a \xrightarrow{\{|0,0\rangle\}} \begin{pmatrix} 0 \end{pmatrix}\,, \ a = 1,2,3\,. \tag{62}$$

$\underline{\mathcal{H}_{\frac{1}{2}}}$

It is two dimensional with basis vectors $\{|\frac{1}{2},\frac{1}{2}\rangle, |\frac{1}{2},-\frac{1}{2}\rangle\}$. The operators are now 2×2 matrices:

$$\begin{aligned} \widehat{J}^2 & \xrightarrow{\{|\frac{1}{2},\frac{1}{2}\rangle,|\frac{1}{2},-\frac{1}{2}\rangle\}} \begin{pmatrix} \frac{1}{2}(\frac{1}{2}+1) & 0 \\ 0 & \frac{1}{2}(\frac{1}{2}+1) \end{pmatrix} \\ \widehat{J}_3 & \xrightarrow{\{|\frac{1}{2},\frac{1}{2}\rangle,|\frac{1}{2},-\frac{1}{2}\rangle\}} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} = \frac{\sigma_3}{2} \\ \widehat{J}_1 & \xrightarrow{\{|\frac{1}{2},\frac{1}{2}\rangle,|\frac{1}{2},-\frac{1}{2}\rangle\}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\sigma_1}{2}\,, \quad \widehat{J}_2 \xrightarrow{\{|\frac{1}{2},\frac{1}{2}\rangle,|\frac{1}{2},-\frac{1}{2}\rangle\}} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \frac{\sigma_2}{2} \end{aligned} \tag{63}$$

In the problem set 1 we saw that the Pauli matrices satisfy the following commutation relations

$$[\frac{\sigma_a}{2}, \frac{\sigma_b}{2}] = i\epsilon_{abc}\frac{\sigma_c}{2}$$

$\underline{\mathcal{H}_1}$

It is three dimensional with basis vectors $\{|1,1\rangle, |1,0\rangle, |1,-1\rangle\}$. The operators are now 3×3 matrices:

$$\widehat{J}^2 \quad \xrightarrow{\{|1,1\rangle,|1,0\rangle,|1,-1\rangle\}} \begin{pmatrix} 1(1+1) & 0 & 0 \\ 0 & 1(1+1) & 0 \\ 0 & 0 & 1(1+1) \end{pmatrix} \tag{64}$$

$$\widehat{J}_3 \quad \xrightarrow{\{|1,1\rangle,|1,0\rangle,|1,-1\rangle\}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\widehat{J}_1 \quad \xrightarrow{\{|1,1\rangle,|1,0\rangle,|1,-1\rangle\}} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \widehat{J}_2 \xrightarrow{\{|1,1\rangle,|1,0\rangle,|1,-1\rangle\}} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$\underline{\mathcal{H}_{\frac{3}{2}}}$

It is four dimensional with basis vectors $\{|\frac{3}{2},\frac{3}{2}\rangle, |\frac{3}{2},\frac{1}{2}\rangle, |\frac{3}{2},-\frac{1}{2}\rangle, |\frac{3}{2},\frac{3}{2}\}$. The operators are now 4×4 matrices

$$\widehat{J}^2 \quad \xrightarrow{\{|\frac{3}{2},\frac{3}{2}\rangle,|\frac{3}{2},\frac{1}{2}\rangle,|\frac{3}{2},-\frac{1}{2}\rangle,|\frac{3}{2},\frac{3}{2}\}} \begin{pmatrix} \frac{3}{2}(\frac{3}{2}+1) & 0 & 0 & 0 \\ 0 & \frac{3}{2}(\frac{3}{2}+1) & 0 & 0 \\ 0 & 0 & \frac{3}{2}(\frac{3}{2}+1) & 0 \\ 0 & 0 & 0 & \frac{3}{2}(\frac{3}{2}+1) \end{pmatrix}$$

$$\widehat{J}_3 \quad \xrightarrow{\{|\frac{3}{2},\frac{3}{2}\rangle,|\frac{3}{2},\frac{1}{2}\rangle,|\frac{3}{2},-\frac{1}{2}\rangle,|\frac{3}{2},\frac{3}{2}\}} \begin{pmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix}$$

$$\widehat{J}_1 \quad \xrightarrow{\{|\frac{3}{2},\frac{3}{2}\rangle,|\frac{3}{2},\frac{1}{2}\rangle,|\frac{3}{2},-\frac{1}{2}\rangle,|\frac{3}{2},\frac{3}{2}\}} \frac{1}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix},$$

$$\widehat{J}_2 \quad \xrightarrow{\{|\frac{3}{2},\frac{3}{2}\rangle,|\frac{3}{2},\frac{1}{2}\rangle,|\frac{3}{2},-\frac{1}{2}\rangle,|\frac{3}{2},\frac{3}{2}\}} \frac{1}{2} \begin{pmatrix} 0 & -i\sqrt{3} & 0 & 0 \\ i\sqrt{3} & 0 & -2i & 0 \\ 0 & 2i & 0 & -i\sqrt{3} \\ 0 & 0 & i\sqrt{3} & 0 \end{pmatrix}.$$

Notice that in each of the above case the matrices satisfy the same commutation relation and that not more than one matrix is diagonal (since otherwise commutation relation will not be satisfied).

The components of the orbital angular momentum are the generators of rotations around the co-ordinate axis. Hence a rotation of 2π around any axis should be represented by identity matrix. However, notice that

$$\exp\Big(i\theta\, J_3\Big) = \begin{pmatrix} e^{i\theta\, j} & 0 & \cdots & 0 \\ 0 & e^{i\theta(j-1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & e^{-i\theta\, j} \end{pmatrix} \tag{65}$$

Therefore

$$\exp\Big(i\, 2\pi\, J_3\Big) = \begin{cases} \mathbb{1}, & j = 0, 1, 2, 3, \cdots \\ -\mathbb{1}, & j = \frac{1}{2}. \frac{3}{2}, \frac{5}{2}, \cdots \end{cases} \tag{66}$$

This implies that orbital angular momentum correspond to integer j values only. Thus half-integer values represent angular momentum which is not orbital and it is called the spin angular momentum.

Exercise 3: The operator $\hat{S}_{\vec{n}}$ of the spin projection on an arbitrary direction $\vec{n}$ is defined as $\vec{S} \cdot \vec{n}$, Where $\vec{S} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$ is the spin angular momentum operator and $\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is a unit vector.
(a). Find the eigenkets $|\frac{1}{2}, m\rangle$ of the operator $\hat{S}_n = \vec{S} \cdot \vec{n}$ for a spin $\frac{1}{2}$ particle.
(b). A particle of spin $\frac{1}{2}$ is prepared in an eigenstate of $\vec{S} \cdot \vec{n}_0$ with the eigenvalue $+\frac{\hbar}{2}$, where $\vec{n}_0 = (\sin\theta, 0, \cos\theta)$. We denote this state by $|\frac{1}{2}, +\frac{1}{2}\rangle_{\vec{n}_0}$.
Suppose s_x, the eigenvalue of $\hat{S}_x$, is measured. What is the probability of obtaining $+\frac{\hbar}{2}$, i.e. finding the particle in the state $|\frac{1}{2}, +\frac{1}{2}\rangle_{\hat{i}}$?
(c). Calculate $\langle(\hat{S}_x - \langle\hat{S}_x\rangle)^2\rangle$ for the state of part (b).

Exercise 4: A beam of spin $\frac{1}{2}$ atoms goes through a series of Stern-Gerlach-type magnets with the following setup.
The first magnet is oriented in the $+z$ direction and the $-\frac{\hbar}{2}$ component of the beam is directed into a beam dump (i.e., stopped). The second magnet is aligned at an angle θ with respect to the z-axis and the $-\frac{\hbar}{2}$ component is dumped (i.e., stopped). The third magnet is oriented in the $+z$ direction and the $+\frac{\hbar}{2}$ component is dumped (i.e., stopped).
(a). What is the intensity of the final $s_z = -\frac{\hbar}{2}$ beam with respect to the initial beam intensity?
(b). How must the second magnet be oriented in order to maximize the intensity of the final $s_z = -\frac{\hbar}{2}$ beam?

Consider two non-interacting systems with angular momentum quantum numbers j_1 and j_2. The Hilbert space of the two systems is denoted by $\mathcal{H}_{j_1}$ and $\mathcal{H}_{j_2}$ (we only consider the angular momentum quantum numbers and ignore possible others). The Hilbert space of the total system, denoted by $\mathcal{H}$, is the tensor product of $\mathcal{H}_{j_1}$ and $\mathcal{H}_{j_2}$. The total angular momentum is given by:

$$\vec{J} = \vec{J}_1 \otimes 1 + 1 \otimes \vec{J}_2 \tag{67}$$

Components of $\vec{J}$ act on $\mathcal{H} = \mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2}$. We will usually not use the tensor product notation and just write the above as $\vec{J} = \vec{J}_1 + \vec{J}_2$ with the understanding that operators labeled by 1 only act on states in $\mathcal{H}_{j_1}$ and operators labeled by 2 only act on states in $\mathcal{H}_{j_2}$. The components of the total angular momentum also satisfy the angular momentum commutation relation:

$$[J_a, J_b] = i\epsilon_{abc}J_c \tag{68}$$

Now we have eight operators acting on $\mathcal{H}$:

$$\{J_{1,x}, J_{1,y}, J_{1,z}, J_1^2, J_{2,x}, J_{2,y}, J_{2,z}, J_2^2\} \tag{69}$$

Of these eight operators we can find four which are mutaully commuting and give us the largest set of commuting operators:

$$\{J_1^2, J_{1,z}, J_2^2, J_{2,z}\} \tag{70}$$

Another combination of four operators which commute are:

$$\{J^2, J_z, J_1^2, J_2^2\} \tag{71}$$

we will label the corresponding eigenstates by the eigenvalues of the operators:

$$\begin{aligned} &\{J_1^2, J_{1,z}, J_2^2, J_{2,z}\} \mapsto |j_1, m_1, j_2, m_2\rangle = |j_1, m_1\rangle \otimes |j_2, m_2\rangle \\ &\{J^2, J_z, J_1^2, J_2^2\} \mapsto |j, m, j_1, j_2\rangle \end{aligned} \tag{72}$$

Since j_1, j_2 are fixed we will simplify the above notation for the eigenstates and simply write

$$|j_1, m_1, j_2, m_2\rangle = |m_1, m_2\rangle \tag{73}$$

$$|j, m, j_1, j_2\rangle = |j, m\rangle \tag{74}$$

It is clear that $\dim_{\mathbb{C}}\mathcal{H} = (2j_1 + 1)(2j_2 + 1)$ and 73 gives two different basis of $\mathcal{H}$:

$$|j, m\rangle = C^{j,m}_{m_1\, m_2}|m_1\, m_2\rangle \tag{75}$$

The coefficients $C^{j,m}_{m_1\,m_2}$ are called the Clebsch-Gordon coefficients.

Example of two spin-$\frac{1}{2}$ particles: Before discussing the general case lets consider the case of two spin $\frac{1}{2}$ particles. In this case the Hilbert space $\mathcal{H}$ is 4 dimensional. The basis $|m_1\,m_2\rangle$ states are:

$$|\frac{1}{2}\,\frac{1}{2}\rangle \equiv |\uparrow\uparrow\rangle \;\mapsto\; \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$|\frac{1}{2}\,-\frac{1}{2}\rangle \equiv |\uparrow\downarrow\rangle \;\mapsto\; \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|-\frac{1}{2}\,\frac{1}{2}\rangle \equiv |\downarrow\uparrow\rangle \;\mapsto\; \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$|-\frac{1}{2}\,-\frac{1}{2}\rangle \equiv |\downarrow\downarrow\rangle \;\mapsto\; \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The above are not the eigenstates of J^2 and J_z. A linear combination of the above are the eigenstates of J^2 and J_z:

$$\begin{aligned} |1,1\rangle &= |\uparrow\uparrow\rangle \\ |1,0\rangle &= \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} \\ |1,-1\rangle &= |\downarrow\downarrow\rangle \\ |0,0\rangle &= \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} \end{aligned}$$

Thus the total angular momentum quantum number takes value $j = 0$ and $j = 1$ each with multiplicity 1:

$$\mathcal{H}_{\frac{1}{2}} \otimes \mathcal{H}_{\frac{1}{2}} = \mathcal{H}_0 \oplus \mathcal{H}_1$$

Two particles with spin j_1 and j_2: Let us consider two particles with spin j_1 and j_2 and corresponding Hilbert spaces $\mathcal{H}_{j_1}$ and $\mathcal{H}_{j_2}$. In this case we would like to determine the possible values of the total angular momentum quantum number. Suppose that in the total angular momentum takes the value j with multiplicity N_j i.e.,

$$\begin{aligned}
\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} &= \sum_{j=0,\frac{1}{2},1,\cdots} N_j \; \mathcal{H}_j \qquad (76)\\
N_j &= \#\text{ of times } \mathcal{H}_j \text{ occurs in the product}
\end{aligned}$$

Since

$$J_z = J_{z\,1} \otimes \mathbb{1} + \mathbb{1} \otimes J_{z\,2} \qquad (77)$$

therefore the state $|m_1\, m_2\rangle$ is an eigenstate of J_z with eigenvalue $m_1 + m_2$. let

$$n(m) = \#\text{ of states with } J_z \text{ eigenvalue equal to } m \qquad (78)$$

Since for a angular momentum quantum number j the values of m go from $-j$ to $+j$ in steps on 1 therefore

$$\begin{aligned}
n(0) &= N_0 + N_1 + N_2 + N_3 + \cdots\\
n(\frac{1}{2}) &= N_{\frac{1}{2}} + N_{\frac{3}{2}} + \cdots\\
n(1) &= N_1 + N_2 + N_3 + \cdots\\
n(\frac{3}{2}) &= N_{\frac{3}{2}} + N_{\frac{5}{2}} + \cdots\\
&\vdots\\
\Longrightarrow N_j &= n(j) - n(j+1)
\end{aligned}$$

We can construct a generating function for N_j ($S = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots\}$)

$$\sum_{j\in S}^{\infty} N_j q^{2j} = \sum_{j\in S} (n(j) - n(j+1)) q^{2j} = n(0)\, q^{-2} + n(\frac{1}{2})\, q^{-1} + (1 - q^{-2}) \sum_{j\in S} n(j) q^{2j} \qquad (79)$$

Notice that

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{H}}\, q^{2J_z} &= \sum_{m\in\{\cdots,-2,-\frac{3}{2},-\frac{1}{2},0,\frac{1}{2},1,\frac{3}{2},2,\cdots\}} n(m)\, q^{2\,m} \qquad (80)\\
&= \sum_{m\in S} n(m)\, q^{2\,m} + \sum_{m\in\{\cdots,-\frac{3}{2},-\frac{1}{2}\}} n(m)\, q^{2\,m}\\
\sum_{m\in S} n(m)\, q^{2\,m} &= \Big[\mathrm{Tr}_{\mathcal{H}}\, q^{2J_z}\Big]_+
\end{aligned}$$

where $[\cdots]_+$ indicates that only non-negative powers of q are kept. From Eq(79) it follows that

$$\begin{aligned}\sum_{j\in S}^{\infty} N_j q^{2j} &= n(0)\, q^{-2} + n(\frac{1}{2})\, q^{-1} + (1-q^{-2}) \sum_{j\in S} n(j) q^{2j} \\ &= n(0)\, q^{-2} + n(\frac{1}{2})\, q^{-1} + (1-q^{-2}) \Big[\mathrm{Tr}_{\mathcal{H}}\, q^{2J_z}\Big]_+ \\ &= n(0)\, q^{-2} + n(\frac{1}{2})\, q^{-1} + q^{-2}(q^2-1) \Big[\mathrm{Tr}_{\mathcal{H}}\, q^{2J_z}\Big]_+ \\ &= q^{-2}\Big[(q^2-1)\,\mathrm{Tr}_{\mathcal{H}}\, q^{2J_z}\Big]_+\end{aligned} \tag{81}$$

$$\begin{aligned}(q^2-1)\mathrm{Tr}_{\mathcal{H}}\, q^{2J_z} &= (q^2-1)\mathrm{Tr}_{\mathcal{H}_{j_1}} q^{2\,J_{1,z}}\, \mathrm{Tr}_{\mathcal{H}_{j_2}} q^{2\,J_{2,z}} \\ &= (q^2-1)(q^{-2j_1} + \cdots + q^{+2j_1})(q^{-2j_2} + \cdots + q^{+2j_2}) \\ &= (q^2-1) q^{-2j_1-2j_2}\, \frac{1-q^{4j_1+2}}{1-q^2} \frac{1-q^{4j_2+2}}{1-q^2} \\ &= q^{-2j_1-2j_2} \frac{q^{4j_1+2}-1}{1-q^2}\, (1-q^{4j_2+2})\end{aligned} \tag{82}$$

It is easy to find the non-negative powers in the above equation assuming $j_1 \geq j_2$:

$$\begin{aligned}q^{-2}\Big[(q^2-1)\,\mathrm{Tr}_{\mathcal{H}}\, q^{2J_z}\Big]_+ &= q^{-2}\Big(q^{2(j_1-j_2)+2} + q^{2(j_1-j_2)+4} + \cdots + q^{2(j_1+j_2)+2}\Big) \\ &= q^{2(j_1-j_2)} + q^{2(j_1-j_2)+2} + \cdots + q^{2(j_1+j_2)}\end{aligned} \tag{83}$$

Threfore

$$\boxed{\sum_{j\in S}^{\infty} N_j q^{2j} = q^{2(j_1-j_2)} + q^{2(j_1-j_2)+2} + \cdots + q^{2(j_1+j_2)}}$$

Thus in the total angular momentum quantum number j takes values from $|j_1 - j_2|$ to $(j_1 + j_2)$ with multiplicity 1.

Exercise 5: Consider the 4 state system consisting of two spin-$\frac{1}{2}$ particles. The Hilbert space of the system is spanned by the 4 orthonormal states:

$$|\uparrow\uparrow\rangle \equiv |\uparrow\rangle_1|\uparrow\rangle_2 \quad |\uparrow\downarrow\rangle \equiv |\uparrow\rangle_1|\downarrow\rangle_2 \quad |\downarrow\uparrow\rangle \equiv |\downarrow\rangle_1|\uparrow\rangle_2 \quad |\downarrow\downarrow\rangle \equiv |\downarrow\rangle_1|\downarrow\rangle_2$$

where the arrows refer to the direction of the spin along the z-axis and the subscript 1 and 2 refer to the particle. Suppose that the Hamiltonian of this system is given by

$$H = \gamma \left(S_{1,z} + S_{2,z}\right) + \frac{\gamma}{\hbar}\vec{S}_1 \cdot \vec{S}_2 \,.$$

a. Write the above Hamiltonian in terms $S_{1,\pm}, S_{1,z}, S_{2,\pm}, S_{2,z}$.
b. Using the form of the Hamiltonian found in part (a) find the matrix of H in the basis given above.
c. Write the Hamiltonian in terms of the $\vec{S}_{total}$ where $\vec{S}_{total} = \vec{S}_1 + \vec{S}_2$.
d. Find the energies and stationary states of the Hamiltonian.
e. If the system is in the state $|\uparrow\downarrow\rangle$ at time $t = 0$ what is the probability of finding the system in the singlet state at time t.

Exercise 6: Consider a system of three particles. Particle 1 has spin $\frac{1}{2}$, particle 2 has spin $\frac{1}{2}$ and particle 3 has spin 1. This system has 12 states:

$$\begin{array}{ccc} |\uparrow\uparrow 1\rangle & |\uparrow\uparrow 0\rangle & |\uparrow\uparrow -1\rangle \\ |\uparrow\downarrow 1\rangle & |\uparrow\downarrow 0\rangle & |\uparrow\downarrow -1\rangle \\ |\downarrow\uparrow 1\rangle & |\downarrow\uparrow 0\rangle & |\downarrow\uparrow -1\rangle \\ |\downarrow\downarrow 1\rangle & |\downarrow\downarrow 0\rangle & |\downarrow\downarrow -1\rangle \end{array}$$

a. What are the possible eigenvalues of J^2_{total}?
b. For each of the eigenvalues found in part (a) what are possible eigenvalues of $J_{total,z}$.
c. Determine $(J_{1,x} + J_{2,x})^2\,|\uparrow\uparrow 1\rangle$ and $J_{1,x}J_{3,y}\,|\uparrow\uparrow 1\rangle$.
d. Write down the normalized state with total angular momentum eigenvalue 0 in terms of the individual spin states given above.

The set of 4×4 matrices which preserve the quadratic form

$$c^2 t^2 - x^2 - y^2 - z^2$$

form a group known as the Lorentz group $O(1,3)$. Since

$$c^2 t^2 - x^2 - y^2 - z^2 = \begin{pmatrix} ct & x & y & z \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix},$$

therefore $g \in O(1,3)$ is such that

$$g^T \eta \, g = \eta$$

This implies that det$(g) = \pm 1$ and therefore $O(1,3)$ is not connected [3]. The identity of the group is the 4×4 identity matrix and belongs to the component with determinant $+1$, this component is denoted with $SO(1,3)$. There are two important transformations which are not in $SO(1,3)$:
Time Reversal T:

$$T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{det}(T) = -1 \tag{84}$$

Parity P:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \text{det}(P) = -1 \tag{85}$$

DIMENSION: To determine the dimension of the group notice that $g^T \eta \, g$ is a symmetric matrix therefore has 10 independent components which are quadratic functions of 16 components of g. The condition $g^T \eta \, g = \eta$ therefore imposes 10 constraints on 16 components of g leaving 6 independent components. Therefore the Lorentz group has dimension 6 and $SO(1,3)$ is generated by 6 generators. Among these 6 generators 3 generate rotation around x, y and z axis. The remaining 3

[3] A set is connected if two points of the set can be connected by a continuous curve. In this case the group has two connected components one with determinant equal to $+1$ and the other with determinant equal to -1.

generate boosts along the x, y and z axis. The component with $g \in O(1,3)$ such that det$(g) = -1$ is given by including the discrete transformations T, P.

Generators: The generators of $SO(1,3)$ are[4]

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \; J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{86}$$

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

These generators satisfy the following commutation relations

$$\begin{aligned} [J_a, J_b] &= \epsilon_{abc} J_c \\ [K_a, K_b] &= -\epsilon_{abc} J_c \\ [K_a, J_b] &= \epsilon_{abc} K_c \end{aligned} \tag{87}$$

If we define new generators $A_a = \frac{K_a + iJ_a}{2}$ and $B_a = \frac{-K_a + iJ_a}{2}$ then

$$[A_a, A_b] = i\epsilon_{abc} A_c \,, \; [B_a, B_b] = i\epsilon_{abc} B_c \,, \; [A_a, B_b] = 0 \tag{88}$$

Thus the new generators A_a and B_b each satisfy the angular momentum commutation relation and commute with each other. Thus we can use the result of the angular momentum commutation relation derived earlier and label the states with two angular momentum quantum numbers, one corresponding to A^2, A_3 and other corresponding to B^2, B_3, j_1 and j_2. Thus representations of the Lorentz group are labeled by two quantum numbers (j_1, j_2) with $j_{1,2} \in \{0, , \frac{1}{2}, 1, \frac{3}{2}, \cdots\}$.

- $(j_1, j_2) = (0, 0)$ is the Lorentz scalar
- $(j_1, j_2) = (\frac{1}{2}, 0)$ is the chiral 2-component spinor
- $(j_1, j_2) = (0, \frac{1}{2})$ is also chiral 2-component spinor
- $(j_1, j_2) = (\frac{1}{2}, \frac{1}{2})$ is the 4-vector
- $(j_1, j_2) = (1, 0)$ is the self-dual 2-form, $F^+_{\mu\nu}$
- $(j_1, j_2) = (0, 1)$ is the antiself-dual 2-form $F^-_{\mu\nu}$

Spinor representation: Chiral, Dirac and Majorana

Chiral 2-component spinor $(\frac{1}{2}, 0)$ transform in an irreducible representation of the Lorentz group. Acting on this 2-component spinor

$$A^a = \frac{\sigma^a}{2} \,, \quad B^a = 0 \;\Rightarrow\; J_a = -i\frac{\sigma_a}{2} \,, \; K_a = \frac{\sigma_a}{2}$$

[4]We denote the rotation generators by $J_{1,2,3}$ and boost generators by $K_{1,2,3}$.

$$
\begin{aligned}
\psi_L &= \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \mapsto_{\text{rotation}} \; e^{-i\theta\,\hat{n}\cdot\frac{\vec{\sigma}}{2}} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \\
\psi_L &= \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \mapsto_{\text{boost}} \; e^{\beta\,\hat{n}\cdot\frac{\vec{\sigma}}{2}} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}
\end{aligned} \tag{89}
$$

Chiral 2-component spinor $(0, \frac{1}{2})$ also transform in an irreducible representation of the Lorentz group. Acting on this 2-component spinor

$$
A^a = 0\,, \quad B^a = \frac{\sigma^a}{2} \;\Rightarrow\; J_a = -i\frac{\sigma_a}{2}\,,\; K_a = -\frac{\sigma_a}{2}
$$

$$
\begin{aligned}
\psi_R &= \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \mapsto_{\text{rotation}} \; e^{-i\theta\,\hat{n}\cdot\frac{\vec{\sigma}}{2}} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \\
\psi_R &= \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \mapsto_{\text{boost}} \; e^{-\beta\,\hat{n}\cdot\frac{\vec{\sigma}}{2}} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}
\end{aligned} \tag{90}
$$

The Dirac spinor ψ_D transforms in $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation of the Lorentz group which is a reducible representation:

$$
\begin{aligned}
\psi_D &= \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \mapsto_{\text{rotation}} \begin{pmatrix} e^{-i\theta\,\hat{n}\cdot\frac{\vec{\sigma}}{2}} & 0 \\ 0 & e^{-i\theta\,\hat{n}\cdot\frac{\vec{\sigma}}{2}} \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \\
\psi_D &= \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \mapsto_{\text{boost}} \begin{pmatrix} e^{\beta\,\hat{n}\cdot\frac{\vec{\sigma}}{2}} & 0 \\ 0 & e^{-\beta\,\hat{n}\cdot\frac{\vec{\sigma}}{2}} \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}
\end{aligned} \tag{91}
$$

Parity:

$$
\psi(\mathbf{r}, t) \mapsto W\,\psi(-\mathbf{r}, t) \tag{92}
$$

where W is a 4×4 matrix representing action of parity on ψ. Since $P^2 = 1$ therefore $W^2 = 1$. Also P commutes with the generators of rotation but anti-commutes with the generators of boost. This implies that

$$
\begin{aligned}
W\,S_{rot}(\vec{n})\,W^{-1} &= S_{rot}(\vec{n}) \\
W\,S_{boost}(\vec{n})\,W^{-1} &= S_{boost}(-\vec{n})
\end{aligned}
$$

It is easy to see that

$$
W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \gamma^0
$$

Thus under parity

$$
\psi(\mathbf{r}, t) \mapsto \gamma^0\,\psi(-\mathbf{r}, t) \tag{93}
$$